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**LETTER TO THE EDITOR**

# Evolutionary quantum game

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**Abstract**

We present a study of a dynamical quantum game. Each agent has a 'memory' of her performance over the previous $m$ timesteps, and her strategy can evolve in time. The game exhibits distinct regimes of optimality. For small $m$ the classical game performs better, while for intermediate $m$ the relative performance depends on whether the source of qubits is 'corrupt'. For large $m$, the quantum players dramatically outperform the classical players by 'freezing' the game into high-performing attractors in which evolution ceases.

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The new field of quantum games is attracting significant interest [1–4]. Benjamin and Hayden [5] created a Prisoner's Dilemma-like quantum game for $N = 3$ players with a high-payoff 'coherent quantum equilibrium' (CQE). Johnson [6] showed that this quantum advantage can become a disadvantage when the game's external qubit source is corrupted by a noisy 'demon'. So far, all studies have been restricted to static games.

Here we introduce an iterated version of the game, where players (agents) may modify their strategies based on information from the past, i.e. they 'learn' from their mistakes [7]. This evolutionary game produces highly non-trivial dynamics in both quantum and classical regimes, and represents the first step toward understanding iterated games employing temporal quantum coherence. Agents are provided with the minimum resources necessary for adaptability—specifically, each agent posesses a measure of her past success through a parameter $\$_m$ whose value reflects the payouts from recent rounds of the game. The fixed 'memory' parameter $m$ effectively governs the number of rounds upon which \$ depends (see later equation (1)), and turns out to be fundamental for deciding the relative superiority of the quantum and classical games. For small memory $m$, the classical game is superior (in the sense that the average payout per player is higher). For intermediate $m$, relative superiority is determined by the reliability of the external qubit source, while for large $m$ the quantum game is radically superior due to evolutionary 'freezing' into a high-paying attractor.

We briefly review the static version of the game [5, 6]. The basic idea is to establish the flow of information in a game, and then investigate the consequences of quantizing this

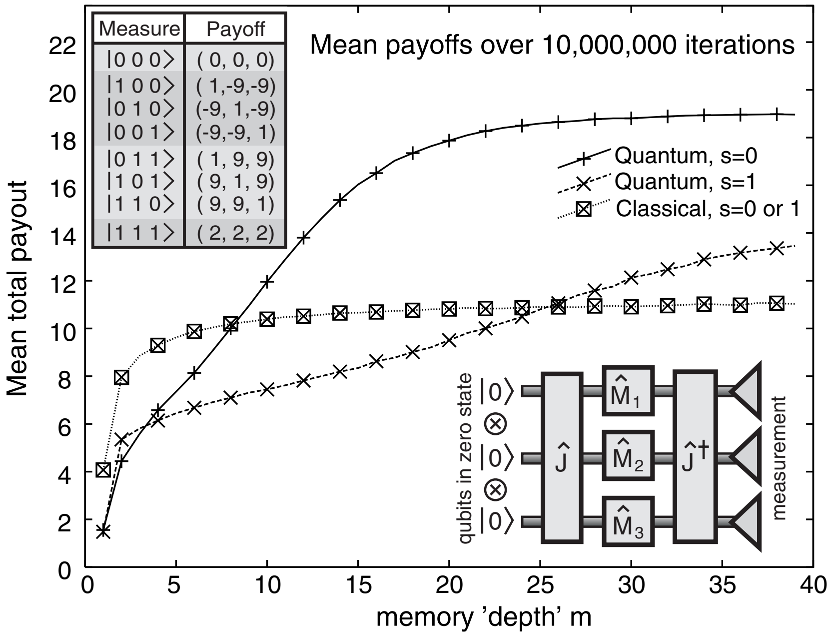


**Figure 1.** Mean total payout per turn as a function of the memory size $m$. Curve with crosses in boxes: classical game with input qubit source 0 or 1. Curves with crosses: quantum games with input qubit source $|0\rangle$ and $|1\rangle$ respectively. Upper inset: Payoff table for our three-player game. Lower inset: circuit diagram showing flow of information in the general quantum game. When the $J$ gates are removed, the game becomes classical.

information. In the classical static game, each of the $N = 3$ players receives a bit in the zero state, and chooses either to 'flip', or 'not flip' this bit prior to returning it. The three bits are then measured, and the payoff to the players is determined by the table shown in figure 1. This payoff table is such that each player's best choice of strategy is simply to choose to 'flip' *regardless* of what any other player may be doing. Given that all players realize this, the inevitable outcome is that all players choose 'flip'—this is the game's 'dominant strategy equilibrium' (DSE). The final measured state is therefore 111, and the corresponding payout to each player is just two points. This result arises despite the existence of cooperative strategies, such as each player choosing 'flip' with probability 80%, which would yield a higher (expected) payoff to all players: the problem is that any given player is better off switching strategy to 'definitely flip', hence this cooperative strategy profile does not occur. This is exactly the same defection problem seen in the famous game Prisoner's Dilemma. The quantization process [5] involves: (a) generalizing each bit to a qubit $\alpha|0\rangle + \beta|1\rangle$, (b) generalizing the moves available to the players from 'flip'/'don't flip' to a set of quantum operations, and (c) introducing 'gates' to entangle the qubits—specifically the entangling gate is $\hat{J} = \frac{1}{\sqrt{2}}(\hat{I}^{\otimes 3} + i\hat{F}^{\otimes 3})$ where $\hat{F} = \hat{\sigma}_x$. Without entanglement, the quantum game is trivially equivalent to the classical variant [5]; therefore we can make a correspondence between the quantum and classical games simply by removing the $J$ gates. The lower inset in figure 1 shows the flow of information in the quantum game. In general the actions of the players in the quantum game will result in a final state which is a superposition. Measurement then collapses this state to one of the classical outcomes, and the payoff is then determined from the table (see figure 1) as in the classical game. Among the conclusions of [5] was the discovery that the quantum players could escape the DSE that

Classical

| Class | p=0 | p=1/2 | p=1 | Average |
|---|---|---|---|---|
| i) | - | aaa(1/2)[1/2]1/2 | - | (1/2)[1/2] |
| ii) | a(21/4)[-17/4]1/2 | aa(3/4)[1/4]1/2 | - | (9/4)[-5/4] |
| iii) | aa(11/2)[-9/2]1/2 | a(3/2)[1/2]1 | - | (25/6)[-17/6] |
| iv) | aaa(2)[0]1 | - | - | (2)[0] |
| v) | - | - | aaa(0)[2]1 | (0)[2] |
| vi) | a(1)[1]1 | - | aa(-9)[9]0 | (-17/3)[19/3] |
| vii) | aa(9)[-9]0 | - | a(1)[1]1 | (19/3)[-17/3] |
| viii) | a(5)[-4]1/2 | a(0)[0]0 | a[-4][5]1/2 | (1/3)[1/3] |
| ix) | - | aa(1/4)[3/4]1/2 | a(-17/4)[21/4]1/2 | (-5/4)[9/4] |
| x) | - | a(1/2)[3/2]1 | aa(-9/2)[11/2]1/2 | (-17/6)[25/6] |

Quantum

| Class | p=0 | p=1/2 | p=1 | Average |
|---|---|---|---|---|
| i) | - | aaa(-15/4)[19/4]1/2 | - | (-15/4)[19/4] |
| ii) | a(-15/4)[19/4]1/2 | aa(-15/4)[19/4]1/2 | - | (-15/4)[19/4] |
| iii) | aa(-7/2)[9/2]1/2 | a(3/2)[1/2]1 | - | (-11/6)[19/6] |
| iv) | aaa(2)[0]1 | - | - | (2)[0] |
| v) | - | - | aaa(0)[2]1 | (0)[2] |
| vi) | a(1)[1]1 | - | aa(-9)[9]0 | (-17/3)[19/3] |
| vii) | aa(9)[-9]0 | - | a(1)[1]1 | (19/3)[-17/3] |
| viii) | a(5)[-4]1/2 | a(9)[-9]0 | a(5)[-4]1/2 | (19/3)[-17/3] |
| ix) | - | aa(19/4)[-15/4]1/2 | a(19/4)[-15/4]1/2 | (19/4)[-15/4] |
| x) | - | a(3/2)[1/2]1 | aa(-7/2)[9/2]1/2 | (-11/6)[19/6] |

**Figure 2.** Average payoffs for classical game (upper) and quantum game (lower). Agents are denoted by ‘a’. $p$ corresponds to the probability of not flipping the input qubit—for the quantum case, $p$ is an operator (see text). Average payoffs for input qubit 0 are shown as (...); those for input 1 are shown as [...]; those for 50:50 mixture of input qubits are shown without parentheses. Final column gives the average payoff per player for a given input qubit.

traps the classical players, and hence outperform the classical players by a considerable factor. In [6] it was shown that this quantum advantage depends on the reliability of the source of qubits: if the source is believed to be generating qubits in the $|0\rangle$ state, but is in fact generating qubits in state $|1\rangle$, then the classical game players will out-perform the quantum ones.

Following the previous literature on iterated classical multi-player games [7], we limit the classical moves to three options: ‘definitely flip’, ‘definitely do not flip’ and ‘flip with probability 1/2’. These options are denoted by setting $p$, the probability of leaving the input qubit unflipped, equal to 0, 1 or 1/2 respectively. This simplification from the full (continuous) range of physically possible $p$ values clarifies the analysis while retaining the basic structure of the game. There are now $3^3 = 27$ possible profiles or ‘configurations’ $(p_1, p_2, p_3)$. These yield ten ‘classes’ each containing $C \geqslant 1$ configurations which are equivalent under interchange of player label [7]. The upper table in figure 2 shows the average payoffs for each configuration class for the classical game. Given that the input is 0, the dominant strategy equilibrium corresponds to all players choosing $p = 0$, i.e. class (iv) in the table. Hence although the continuous-parameter $p$-space has been discretized to only three values, this description includes the fundamental dominant strategy equilibrium.

To maintain a correspondence between the quantum and classical games, we also permit our quantum players just three different moves: $\hat{I}$, $\hat{\sigma}_x$ and $\frac{1}{\sqrt{2}}(\hat{\sigma}_x + \hat{\sigma}_z)$. It was earlier shown that these moves can give rise to superior quantum performance for the static quantum game with reliable qubit source [5]. Moreover when the entangling $J$ gates are removed, making the game classical, these three moves correctly correspond to our allowed classical moves $p = 1, 0, 1/2$ (respectively). Therefore we label the three quantum moves with the same $p$ notation. A fully comprehensive study of the quantum game would also examine the case where the players are permitted a larger set of moves, e.g. one that is closed under composition. However, the aim here is to make a first direct move-for-move comparison between the iterated quantum and classical games, and this necessitates restricting the larger set of quantum options. The resulting table (see figure 2) provides a simple quantum analogue of the classical case.

We assume that in both quantum and classical games, players are unable to communicate between themselves and hence cannot coordinate which player picks which strategy. In the quantum game, this is more critical since the CQE (i.e. the Nash equilibrium given by class (viii) in the lower table of figure 2) involves players using different $p$'s. We will find that the evolutionary aspect of our iterated game allows the quantum players to find cooperative solutions even in the absence of communication, provided that the memory $m$ is not too small.

The agents are allowed to evolve ('mutate') their strategies based on past success, according to the following prescription. After timestep $t - 1$, each player holds a 'moving average' $\$_{m,t-1}$ of her payoffs which is then updated at timestep $t$:

$$\$_{m,t} = \frac{m-1}{m}\$_{m,t-1} + \frac{1}{m}\Delta_t \tag{1}$$

where $\$_{m,t}$ is the updated moving average and $\Delta_t$ is the payoff to the player at turn $t$. The information about previous outcomes therefore has a 'half life' since the contribution of a given round's payoff falls exponentially with successive rounds. When the player's moving average falls below a certain threshold $d$, she mutates—she chooses one of the other two allowed strategies with a 50 : 50 chance. After mutation, the moving average $\$_m$ continues to be updated, but the player enters a 'trial period' of $Y$ turns before considering mutating again. For simplicity, we set $Y = m$ for the remainder of this paper. We also consider fixed external qubit source orientations, although each of these restrictions can be relaxed. We choose a mutation threshold $d = 3.0$ in order to be above the two point maximum possible payoff per player in the classical static game (see payoff table in figure 1). We performed runs with $10^7$ timesteps—this is sufficient for the resulting statistics to be stable to within a few percent.

Figure 1 shows the results for the average payoff per turn for the three agents combined. The classical dominant strategy equilibrium (2, 2, 2) corresponds to a value 6, while the CQE (5, 9, 5) and its permutations correspond to a value 19. Figure 3 shows the corresponding average lifetime of the agents before mutation. Figure 1 shows that for memory $m < 8$, the quantum game players are *worse* off than the classical game players regardless of the input qubit source orientation (0 or 1). (The particular orientation of the fixed source does not affect the results for the classical game.) The classical game players are also less prone to mutation, as seen from figure 3. For $8 < m < 25$, the quantum game players do better than the classical players if the source is 0 (as they believe it is) but do worse if the source is 1 (i.e. the source is 'corrupt' [6]). The same is qualitatively true for the lifetimes (figure 3) but for a smaller range $8 < m < 19$. For $m > 25$, the quantum game players achieve better average payoffs than the classical game players irrespective of the source orientation, but those with the reliable (i.e. non-corrupt) source do significantly better. They eventually 'freeze' into a cooperative state, as explained later.

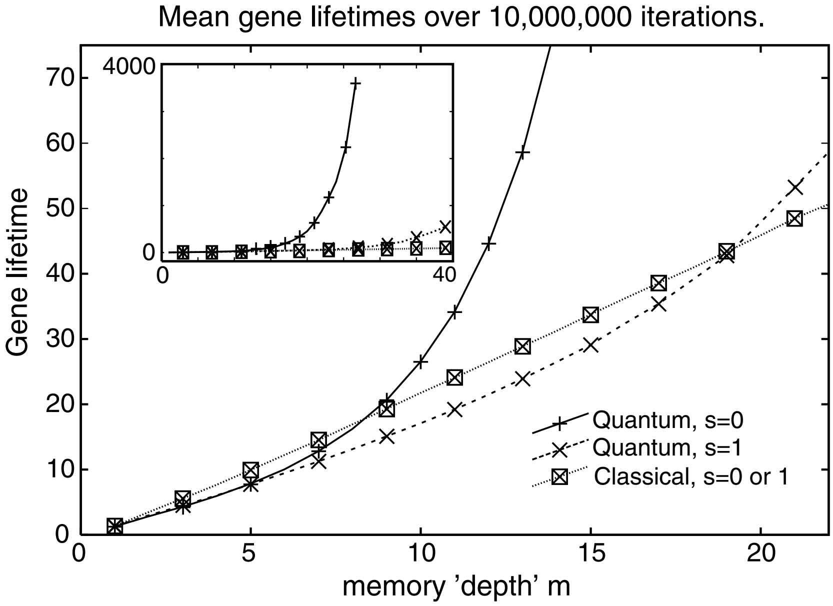


**Figure 3.** Average time agents maintain a given strategy before mutating, as a function of the memory $m$. Inset: Results for a larger $y$-scale, illustrating the 'freezing' of the dynamics in the quantum game with a reliable qubit source ($s = 0$).

Figure 1 reveals an interesting effect within the classical game: for $m > 1$ the classical game players do better than the classical dominant strategy equilibrium value of $2+2+2 = 6$. If we had set the threshold $d \leqslant 2$, the players would quickly 'freeze' into the configuration $p = 0, 0, 0$, which guarantees two points per player per turn. By adopting a higher satisfaction limit, i.e. by being more 'greedy', the classical players introduce instability into the game—although this implies that in any given round at least one player is typically worse off, in the long term the instability actually benefits all the players.

The explanation for the interesting large-$m$ behaviour lies in the cycles through which the game moves on short and long timescales. In general, the behaviour of both classical and quantum systems for larger $m$ is dominated by a small number of possible attractors which trap the system for a certain period, before it jumps into a brief period of turbulence and then ends up in another attractor.

Figure 4 shows the structure of the attractors in the classical and quantum cases. In the classical game, the attractors correspond to two players fixing their strategy as 'definitely flip' ($p = 0$). While they maintain these choices, the third player can never achieve satisfactory payouts, hence the short timescale dynamics correspond to this third player jumping around the three possible $p$ values, constantly mutating. Eventually one of the other two players will encounter a run of losses and mutate her strategy. The game then enters a (poorly performing) turbulent period before settling back into either the previous attractor or one of the two permutation-equivalent cycles. In the quantum case, however, these three short cycles are replaced by six attractors, each a single configuration such as $p = 0, 1, 1/2$. Because all such attractors yield satisfactory expected payoff to all players, the system will remain in an attractor until one player has an exceptionally long run of losses. Figure 3 indicates that the probability of this occurence falls exponentially with $m$, i.e. the system 'freezes' into one configuration profile.

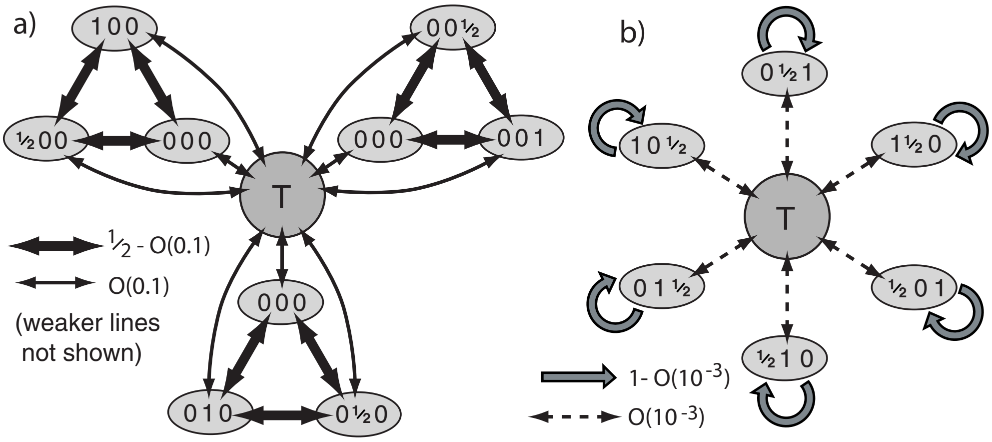


**Figure 4.** Schematic game dynamics showing the attractors which dominate both the classical and quantum games in the large $m$ limit. ($a$) The classical game, and ($b$) the quantum game with a reliable source ($s = 0$). Ellipses represent specific strategy profiles and are labelled by the $p$ values adopted by the three players. The circle labelled $T$ represents the set of all other profiles, which are visited only during brief turbulent periods in the game's dynamics. Arrows indicate the probability that one profile follows another over a period of $m$ rounds. Data are collected over $10^7$ rounds with $m = 30$. In the classical game the profile ($p_1 = 0$, $p_2 = 0$, $p_3 = 0$), corresponding to the dominant strategy equilibrium, occurs in each of the three attractor cycles.

In summary, we have presented the first results for an evolutionary quantum game. The dynamics are non-trivial, even in the present case where the game's history is assumed to be classical in that a measurement is taken at each timestep. The next logical step is to to allow for a superposition (or 'multiverse') of histories over $R > 1$ turns. Because the coherence in the present game need only be maintained for $R = 1$ turns, the present game could be implemented as an algorthim in an elementary quantum computer containing three or more qubits—such an experiment could be performed with existing NMR technologies. More generally, it is not inconceivable that such 'games' are already being played at some microscopic level in the physical world —indeed, it has recently been shown that nature plays classical dominant-strategy games using clones of a virus that infects bacteria [9].

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